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Ladder approximation to fermion quasi-particle interaction for exponentially varying bare potentials

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Abstract. An extension is developed of earlier work on solution of the Bethe–Salpeter equation for the ladder approximation, Γ_L, to the effective interaction in a Fermi liquid. This permits treatment of the case of bare potentials having finite range, with a spatial dependence which is either exponential or is representable as the Laplace transform of another function, and an exponential decay in time. The integral equation is transformed to real space and the kernel replaced by a differential operator, in a generalisation of an approach developed by Hahne, Heiss, and Engelbrecht for interactions depending only on time. This procedure simplifies calculation of the terms in the iterative solution of the integral equation, as is demonstrated by explicit calculation, for several cases, of the first two iterative terms. It is found, in agreement with earlier results, that going from a zero-range or Dirac delta interaction to one of finite range can change markedly the analytic character of Γ_L and its calculation.

1. Introduction

If an atom is introduced into a fermion system at 0 K, it can interact via the bare potential with an atom in the Fermi sea, creating a hole which propagates forward in time, experiencing any number of virtual scattering events with the original atom, until a second atom falls into it. The sum of all Feynman diagrams representing such a sequence of scattering processes, with ingoing four-momenta $p_1-\varepsilon$ and p_2 and outgoing p_1, $p_2-\varepsilon$, as $\varepsilon \to 0$ gives the particle–hole ladder approximation $\Gamma_L(p_1, p_2; p_1, p_2)$ to both the effective interaction and scattering amplitude (Nozières 1964).

The ladder approximation includes only the simplest diagrams but it has been used (Babu and Brown 1973) in conjunction with the contact interaction potential

$$V_c(\boldsymbol{r}, t) = I\delta(t)\delta(\boldsymbol{r}). \tag{1}$$

Equation (1) must be an idealisation of a process which has a very short, but finite, range in time and space. With this in mind we considered (Nettleton 1979) the potential

$$V(\boldsymbol{r}, t) = I\exp(-\beta|t|)\delta(\boldsymbol{r}) \tag{2}$$

and the evaluation of the ladder sum as β becomes very large, which should be a close physical approximation to equation (1).

When we cannot sum the ladder diagrams term by term, we use the fact that the sum is the solution of the Bethe–Salpeter equation (Fetter and Walecka 1971, p 137) which

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for a particle–hole ladder is

$$\Gamma_L(p_1-w, p_2; p_1, p_2-w)$$
$$= U(-w)+\frac{\mathrm{i}}{\hbar(2\pi)^4}\int \mathrm{d}^4q\, U(q-w)G^0(p_1-q)G^0(p_2-q)$$
$$\times \Gamma_L(p_1-q, p_2; p_1, p_2-q) \tag{3}$$

where $U(q)\equiv U(q_0, \boldsymbol{q})$ is the four-dimensional Fourier transform of $V(\boldsymbol{r}, t)$, and

$$G^0(k)\equiv G^0(\boldsymbol{k}, \omega)=\left(\frac{\theta(|\boldsymbol{k}|-k_F)}{\omega-\omega_k+\mathrm{i}\eta}+\frac{\theta(k_F-|\boldsymbol{k}|)}{\omega-\omega_k-\mathrm{i}\eta}\right) \tag{4}$$

is the free-fermion Green function, θ being the Heaviside step function and $\omega_k = \hbar k^2/2m$. We assume $|\boldsymbol{p}_1| = |\boldsymbol{p}_2| = p_F$ which characterises quasi-particles of interest near 0 K. Although the Bethe–Salpeter equation is often encountered (cf Fetter and Walecka 1971), techniques for its solution are in a rudimentary state of development, and none has been developed for potentials having a soft spatial dependence.

The solution of equation (3) for the potential of equation (2) was undertaken previously (Nettleton 1979) with the aid of an approach developed by Hahne *et al* (1977). This procedure first takes the Fourier transform of equation (3) to yield an equation relating the transforms

$$\hat{\Gamma}(y, \boldsymbol{z})\equiv\int \Gamma_L(p_1-q, p_2; p_1, p_2-q)\exp(-\mathrm{i}q_o y)\exp(\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{z})\,\mathrm{d}q_0\,\mathrm{d}\boldsymbol{q} \tag{5a}$$

$$\hat{U}(y, \boldsymbol{z})\equiv\int U(q_0, \boldsymbol{q})\exp(-\mathrm{i}q_0 y)\exp(\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{z})\,\mathrm{d}q_0\,\mathrm{d}\boldsymbol{q}. \tag{5b}$$

The $(y, \boldsymbol{z})$ transform of equation (3) has the form

$$\hat{\Gamma}(y, \boldsymbol{z})=\hat{U}(y, \boldsymbol{z})+\hat{U}(y, \boldsymbol{z})[\mathrm{i}/\hbar(2\pi)^4]\int \mathrm{d}q_0\,\mathrm{d}\boldsymbol{q}\,\exp(-\mathrm{i}q_0 y)\exp(\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{z})$$
$$\times\frac{1}{(q_0-\omega_F+\omega_{\boldsymbol{p}_1-\boldsymbol{q}}\pm\mathrm{i}\eta)(q_0-\omega_F+\omega_{\boldsymbol{p}_2-\boldsymbol{q}}\pm\mathrm{i}\eta)}\Gamma_L(p_1-q, p_2; p_1, p_2-q) \tag{6}$$

where we use the fact that $\boldsymbol{p}_1$, $\boldsymbol{p}_2$ are on the Fermi surface to write $\omega_{\boldsymbol{p}_1}=\omega_F=\omega_{\boldsymbol{p}_2}$, and the upper or lower sign before the infinitesimal is used in the appropriate sub-domain of the $\boldsymbol{q}$ integration.

To solve equation (6) by the method of Hahne *et al* (1977) we replace $q_0\to \mathrm{i}D_y\equiv \mathrm{i}\,\mathrm{d}/\mathrm{d}y$ in the denominator, leading to a differential equation readily solvable in the simple case where V does not depend on $\boldsymbol{r}$. In the more complicated case of equation (2) (Nettleton 1979), the differential operator was an integral over $\boldsymbol{q}$ which led to a difficult contour integration. The latter, in fact, posed a major stumbling block to application of the method and did not encourage the belief that it could readily be generalised to include softer interactions. Here we shall consider potentials having a finite range in space, where we can replace $\boldsymbol{q}$ or, in the case of spherical potentials, q in the denominator of equation (6) by a derivative with respect to $\boldsymbol{z}$ or z respectively. The resulting partial differential equation can be solved iteratively when $V(\boldsymbol{r}, t)$ has the time dependence $\exp(-\beta_1|t|)$.

When the solution to equation (6) has been obtained, the $(y, \boldsymbol{z})$ transform must be inverted by calculating

$$\Gamma_{\mathrm{L}}(p_1, p_2; p_1, p_2) = \frac{1}{(2\pi)^4} \int \hat{\Gamma}(y, \boldsymbol{z})\, \mathrm{d}y\, \mathrm{d}\boldsymbol{z}. \tag{7}$$

A simple case where all the integrals can be performed, yielding a series in negative powers of β_1, will be discussed in § 2. The essential feature of the method is that the space- and time-dependence of V should be exponential, and it will work when $V(\boldsymbol{r}, t)$ is the Laplace transform of another function. Five such cases representing spherical potentials are considered in § 3, where equation (2) reduces in general to an integral over a single variable which must ultimately be performed numerically, although we shall give an analytic evaluation of the integrals for two of the cases considered. These results will be summarised in § 4.

2. Solution for potential damped exponentially in time and space

To the extent that the contact potential, V_c, represents the process of paramagnon exchange, it should be physically reasonable to represent the same process by a potential of very short but finite range. We shall consider here an anisotropic potential, which is unrealistic, but leads to a simple result which can be compared with Γ_{L} calculated from equations (1) and (2). Having illustrated the method for this simple case, we go on in § 3 to take up the spherically symmetric case.

We proceed to discuss the solution of equation (6), assuming

$$V(\boldsymbol{r}, t) = I \exp(-\beta_1|t|) \exp[-(\beta_{21}|X| + \beta_{22}|Y| + \beta_{23}|Z|)] \tag{8}$$

where $\beta_1 > 0 < \beta_{2i}$, and X, Y, Z are the components of $\boldsymbol{r}$. We now have

$$\hat{U}(y, \boldsymbol{z}) = (2\pi)^4 I \exp(-\tilde{\beta}_1 y) \exp(-\tilde{\boldsymbol{\beta}}_2 \cdot \boldsymbol{z}) \tag{9}$$

where $\tilde{\beta}_1 \equiv \beta_1 \operatorname{sgn} y$, and similarly the components of $\tilde{\boldsymbol{\beta}}_2$ are <0 when the corresponding component of $\boldsymbol{z}$ is negative.

To convert equation (6) into a differential equation replace $q_0 \to \mathrm{i}D_y$, $\boldsymbol{q} \to \mathrm{i}\nabla_z$. This permits us to take the G^0 factors out from under the integral sign, provided we assume the y dependence of Γ_{L} to be $\sim\exp(-\tilde{\beta}_1 y)$ so that we can set $\eta = 0$ (cf Nettleton 1979), and equation (6) becomes

$$\hat{\Gamma}(y, \boldsymbol{z}) = \hat{U}(y, \boldsymbol{z}) + \hat{U}(y, \boldsymbol{z})[\mathrm{i}/\hbar(2\pi)^4]\left(\mathrm{i}\mathrm{D}_y + \frac{\mathrm{i}\hbar}{m}\boldsymbol{p}_1 \cdot \nabla_z - \frac{\hbar}{2m}\nabla_z^2\right)^{-1}$$
$$\times \left(\mathrm{i}\mathrm{D}_y + \frac{\mathrm{i}\hbar}{m}\boldsymbol{p}_2 \cdot \nabla_z - \frac{\hbar}{2m}\nabla_z^2\right)^{-1} \hat{\Gamma}(y, \boldsymbol{z}). \tag{10}$$

We try a solution

$$\hat{\Gamma}(y, \boldsymbol{z}) = \sum_{n \geqslant 1} A_n \exp[n(-\tilde{\beta}_1 y - \tilde{\boldsymbol{\beta}}_2 \cdot \boldsymbol{z})] \tag{11}$$

which satisfies equation (10) provided that

$$A_1 = (2\pi)^4 I \tag{12a}$$

$$A_{n+1}=\frac{\mathrm{i}I}{\hbar}\left(-\mathrm{i}n\beta_1-\frac{\mathrm{i}\hbar}{m}n\boldsymbol{p}_1\cdot\tilde{\boldsymbol{\beta}}_2+\frac{\hbar}{2m}(n\beta_2)^2\right)^{-1}\left(-\mathrm{i}n\beta_1-\frac{\mathrm{i}\hbar}{m}n\boldsymbol{p}_2\cdot\tilde{\boldsymbol{\beta}}_2+\frac{\hbar}{2m}(n\beta_2)^2\right)^{-1}\cdot A_n. \tag{12b}$$

From equation (12b), we see the expansion converges if either β_1 or $|\beta_2|$, or both, are sufficiently large. If equations (12a, b) are used to calculate the coefficients in equation (11), we obtain a general solution, which is complicated because the signs of $\tilde{\beta}_1$ and the components of $\tilde{\boldsymbol{\beta}}_2$ vary with the sub-domains of the y and $\boldsymbol{z}$ integration in equation (7). It is more instructive to take $\beta_1 \gg |\boldsymbol{\beta}_2|$, where we obtain

$$\Gamma_{\mathrm{L}}=\frac{16I}{\beta_1\Pi_{i=1}^3\beta_{2i}}\sum_{n\geq 0}\frac{(-1)^n\nu^{2n}}{[(2n)!]^2}\left(\frac{1}{(2n+1)^4}-\frac{\mathrm{i}\nu}{(2n+1)^2(2n+2)^4}\right),\qquad \nu\equiv I/\beta_1^2\hbar. \tag{13}$$

We note that the dependence on $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$ has disappeared, as was found (Nettleton 1979) in the case of equation (2). Also, Γ_{L} is complex, except in the $\beta_1\to\infty$ limit, which also obtains for equation (2) (after correction of an error in equation (11b) of Nettleton 1979) but not for V_{c}.

An alternative approximation suggested by equation (2) is to suppose that $|\boldsymbol{\beta}_2| \gg \beta_1$ and expand in powers of $|\boldsymbol{\beta}_2|^{-1}$.

The first two terms have the form

$$\Gamma_{\mathrm{L}}=\frac{I}{\beta_1\Pi_{j=1}^3\beta_{2j}}\left\{16+\frac{\mathrm{i}I}{\hbar|\boldsymbol{\beta}_2|^4}\left(\frac{2m}{\hbar}\right)^2\left[1-\frac{4}{|\boldsymbol{\beta}_2|^4}\sum_{k=1,2}\sum_{i=1}^{3}\sum_{j=1}^{3}\tfrac{1}{2}(1+\delta_{ij})\right.\right.$$
$$\left.\left.\times(p_{ki}\beta_{2i})(p_{kj}\beta_{2j})\right]+\mathrm{O}(|\boldsymbol{\beta}_2|^{-8})\right\}. \tag{14}$$

As $|\boldsymbol{\beta}_2|\to\infty$ with $I=\mathrm{O}(|\boldsymbol{\beta}_2|^3)$, this does not reduce to Γ_{L} calculated from equation (2), just as equation (2), in the limit $\beta_1\to\infty$ with $I=\mathrm{O}(\beta_1)$, does not reduce to the result of equation (1). This circumstance, that the operation of going to the limit of infinite damping does not commute in general with that of calculating the ladder sum, can also be seen in a straightforward iteration of the Bethe–Salpeter equation, without the differential operator techniques used here.

3. Spherical interaction with spatial dependence representable by Laplace transform

Equations (13) and (14) serve to illustrate our approach to simplifying the iterative solution to the Bethe–Salpeter equation by replacing q_0 and $\boldsymbol{q}$ with differential operators and taking the kernel outside the integral in equation (6). However, when $V(\boldsymbol{r}, t)$ is spherically symmetric, additional problems arise, since it is the dependence of the kernel on q, but not the angular dependence, which can be treated by the methods of § 2. The result is that equation (7) is replaced by an integral which must be taken over the angles $\theta_{\boldsymbol{q}}$, $\varphi_{\boldsymbol{q}}$ as well as over y and $\boldsymbol{z}$. The simplest case of the spherical potential, where both the space- and time-dependence of $V(\boldsymbol{r}, t)$ are exponential, is readily generalised to the case where $V(\boldsymbol{r}, t)$ is a Laplace transform, and we shall consider some of those cases as well in the present section. These solutions may prove useful if we try to include higher-order diagrams by replacing the rungs of the ladder with renormalised interactions, so that $\hat{U}(y, \boldsymbol{z})$ is interpreted as the real-space equivalent of the latter. The methods of this section may also find application to Fermi liquids other than ^{3}He where paramagnon exchange, for which the strongly-damped potentials are a model, may not be dominant.

3.1. General approach

To treat the problem of solving equation (6) for the case of a potential $V(\boldsymbol{r}, t)$ having exponential time decay but a more general r dependence, suppose that

$$\hat{U}(y, \boldsymbol{z}) = I \exp(-\beta_1|y|) \int_0^\infty \psi(p) \exp(-zp)\, \mathrm{d}p, \qquad z \equiv |\boldsymbol{z}|. \tag{15}$$

We are thus assuming that $V(\boldsymbol{r}, t)$ has a Fourier transform and that its spatial dependence has an inverse Laplace transform $\psi(p)$. The exponential z dependence in equation (21) permits us to apply the approach developed in the preceding section, with the added complication that the solution depends on the additional parameter p over which we must integrate.

We make the *ansatz*

$$\hat{\Gamma}(y, \boldsymbol{z}) = \int \sum_{n \geq 1} A_n(\theta, \theta_i, p) \exp(-n\beta_1|y| - pz)\, \mathrm{d}p\, \mathrm{d}\Omega \tag{16}$$

where θ is the angle between $\boldsymbol{q}$ and $\boldsymbol{z}$ and θ_i $(i = 1, 2)$ the angle between $\boldsymbol{q}$ and $\boldsymbol{p}_i$. This will solve equation (6) provided

$$A_1 = \frac{I}{4\pi} \psi(p)$$

$$A_{n+1} = \frac{\mathrm{i}I}{\hbar(2\pi)^4} \int_0^p \psi(p - p') \frac{1}{-\mathrm{i}n\tilde{\beta}_1 - \mathrm{i}\hbar p_1(\cos\theta_1/m\cos\theta)p' + \hbar p'^2/(2m\cos^2\theta)}$$
$$\times \frac{1}{-\mathrm{i}n\tilde{\beta}_1 - \mathrm{i}\hbar p_2(\cos\theta_2/m\cos\theta)p' + \hbar p'^2/(2m\cos^2\theta)} A_n\, \mathrm{d}p' \tag{17}$$

where $\tilde{\beta}_1$ has the sign of y.

From equation (7) we see that the nth order contribution to Γ_{L} is

$$\Gamma_{\mathrm{L}}^{(n)} = \frac{1}{(2\pi)^4} \int A_n(\theta, \theta_i, p) \exp(-n\beta_1 y - pz)\, \mathrm{d}p\, \mathrm{d}\Omega\, \mathrm{d}y\, \mathrm{d}z$$
$$= \frac{2}{n\beta_1(2\pi)^3} \int \frac{1}{p^3}[A_n + \tilde{A}_n]\, \mathrm{d}p\, \mathrm{d}\Omega\, \mathrm{d}x \tag{18}$$

where $\mathrm{d}x = \sin\theta_z\, \mathrm{d}\theta_z$ comes from integrating over the polar angles of $\boldsymbol{z}$, and A_n, $\tilde{A}_n$ differ only in the sign of $\tilde{\beta}_1$. Equation (18) can be valid only if it is legitimate to invert the order of integration and integrate first over z, and indeed the examples chosen below will all have this property.

Evidently the problem of calculating the higher-order A_n by recursive application of equation (17) is extremely complex, and in the present section we shall limit ourselves to $\Gamma_{\mathrm{L}}^{(1)}$ and $\Gamma_{\mathrm{L}}^{(2)}$. If β_1 is large or I small, the sequence of terms, $\Gamma_{\mathrm{L}}^{(n)}$, should decrease rapidly with increasing n, and the first two terms should yield a useful approximation.

The $n=2$ term is

$$\Gamma_L^{(2)} = \frac{\mathrm{i}I^2}{2(2\pi)^8 \hbar\beta_1} \sum_{j=1,-1} \int \mathrm{d}x\, \mathrm{d}\Omega \int_0^\infty \frac{\mathrm{d}p}{p^3} \int_0^p \mathrm{d}p'\, \psi(p-p')\psi(p')$$
$$\times \frac{x^4}{-\mathrm{i}j\beta_1 x^2 - \mathrm{i}(\hbar/m)p_1 \cos\theta_1 x p' + (\hbar/2m)p'^2}$$
$$\times \frac{1}{-\mathrm{i}j\beta_1 x^2 - \mathrm{i}(\hbar/m)p_2 \cos\theta_2 x p' + (\hbar/2m)p'^2} \tag{19}$$

The integrals in equation (19) can be simplified via the transformation

$$\xi = p'/x, \qquad s = p/p'$$

to yield

$$\Gamma_L^{(2)} = -\frac{\mathrm{i}4I^2}{(2\pi)^7 \hbar\beta_1} \int_0^\infty \frac{\mathrm{d}\xi}{\xi^2} \Phi(\xi) \int_1^\infty \frac{\mathrm{d}s}{s^3} \int_0^1 \frac{\mathrm{d}x}{x} \psi(x\xi(s-1))\psi(x\xi). \tag{20}$$

The function $\Phi(\xi)$ comes from the Ω integration of the last two factors in equation (19) which can be effected most easily when $\hbar\xi^2/m\beta_1 \ll 1$ or when $\hbar\xi^2/m\beta_1 \gg 1$, and we give explicitly these two cases. We have (θ_p is the angle between $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$)

$$\beta_1^2 \Phi(\xi) = 1 + (\hbar\xi/m\beta_1)^2 (p_1^2 + p_2^2 + p_1 p_2 \cos\theta_p - \tfrac{3}{4}\xi^2)$$
$$+ \mathrm{O}((\hbar\xi^2/m\beta_1)^3)(\hbar\xi^2/m\beta_1 \ll 1) \tag{21a}$$

$$\Phi(\xi) = (2m/\hbar\xi^2)^2[-1 + \tfrac{1}{3}\xi^{-2}(p_1^2 + p_2^2 + p_1 p_2 \cos\theta_p) + \mathrm{O}(\xi^{-4})](\hbar\xi^2/m\beta_1 \gg 1) \tag{21b}$$

We shall consider several cases in which the s and x integrations in equation (20) can be performed, yielding an expression for $\Gamma_L^{(2)}$ in the form of a single integral over ξ which in general must be evaluated numerically.

3.2. *Special cases*

3.2.1. $\psi(p) = (2\pi)^4 \delta(p - \beta_2)$ (β_2 large and >0).

$$V(\boldsymbol{r}, t) = I \exp(-\beta_1|t| - \beta_2 r),\ r \equiv |\boldsymbol{r}| \tag{22a}$$

$$\Gamma_L^{(1)} + \Gamma_L^{(2)} = \frac{2I}{\beta_1\beta_2^3}\left\{8\pi + \frac{\mathrm{i}I}{8\hbar\beta_2^4}\left(\frac{2m}{\hbar}\right)^2\left[\frac{1}{5} - \frac{8k_F^2}{21\beta_2^2}\left(\frac{4}{3}(1-\pi)\right.\right.\right.$$
$$\left.\left.\left. + P_1(\cos\theta_p) + \tfrac{2}{3}(1-2\pi)P_2(\cos\theta_p)\right)\right]\right\}. \tag{22b}$$

If we compare equation (22*b*) with equation (14), we see that Γ_L is again complex and Re (Γ_L) independent of θ_p. Again it is true that the $\beta_2 \to \infty$ limit of equation (22*b*) does not give the same expression as does equation (1), although the effective interaction becomes real in this limit. The Landau parameters F_1 and F_2 are imaginary.

3.2.2. $\psi(p) = p^{\nu-1} \exp(-a/4p)(a>0, 0 \leqslant \nu \leqslant 2)$:

$$V(\boldsymbol{r}, t) = 2(2\pi)^{-4} I \exp(-\beta_1|t|)(a/4r)^{\nu/2} K_\nu[(ar)^{1/2}] \tag{23a}$$

where K_ν is a modified Bessel function.

$$\Gamma_{\mathrm{L}}^{(1)}=\frac{3I}{\pi^3\beta_1}\left(\frac{4}{a}\right)^{3-\nu}(2-\nu)! \tag{23b}$$

$$\gamma(\xi)=\tfrac{1}{2}\left[1+\frac{1}{2}\left(\frac{a}{4\xi}\right)^2\right]E_1\left(\frac{a}{4\xi}\right)-\frac{1}{4}\left(3+\frac{a}{4\xi}\right)\exp(-a/4\xi) \qquad (\nu=1) \tag{23c}$$

where E_1 is an exponential integral function (Abramowitz and Stegun 1964 p 228).

3.2.3. $\psi(p)=0(0\leqslant p<\alpha_1)$; $\psi(p)=-\psi_0\ (\alpha_1\leqslant p<\alpha_2)$; $\psi(p)=-\psi_1\ (\alpha_2\leqslant p)$:

$$V(\boldsymbol{r},t)=(2\pi)^{-4}I\exp(-\beta_1|t|)r^{-1}[(\psi_0-\psi_1)\exp(-\alpha_2 r)-\psi_0\exp(-\alpha_1 r)] \tag{24a}$$

$$\Gamma_{\mathrm{L}}^{(1)}=(3I/2\pi^3\beta_1)[(\psi_0-\psi_1)\alpha_2^{-2}-\psi_0\alpha_1^{-2}]. \tag{24b}$$

The general expression for $\gamma(\xi)$ is readily obtained, but lengthy, and we specialise to the case $\psi_1=0$, for which

$$\gamma(\xi)=0 \qquad (\xi<\alpha_1) \tag{24c}$$

$$\gamma(\xi)=-\tfrac{1}{2}\psi_0^2\left[\ln\left(\frac{\xi+\alpha_2}{\alpha_1+\alpha_2}\right)-\ln\left(\frac{\xi+\alpha_1}{2\alpha_1}\right)\right.$$
$$\left.+\alpha_2\left(\frac{1}{\xi+\alpha_2}-\frac{1}{\alpha_1+\alpha_2}\right)-\alpha_1\left(\frac{1}{\xi+\alpha_1}-\frac{1}{2\alpha_1}\right)\right] \qquad (\alpha_1\leqslant\xi\leqslant\alpha_2) \tag{24d}$$

$$\gamma(\xi)=-\tfrac{1}{2}\psi_0^2\ln\left[\frac{4\ {}_1\alpha_2}{(\alpha_1+\alpha_2)^2}\right] \qquad (\xi>\alpha_2). \tag{24e}$$

To use this in equation (20) we need to approximate $\Phi(\xi)$. We shall assume that $\hbar\alpha_1^2/m\beta_1\gg1$ which corresponds to strong spatial damping, consistent with the model of the preceding section, and thus we adopt equation (21b). The integral in equation (20) can now readily be evaluated with the aid of expansions in partial fractions, and we obtain

$$\Gamma_{\mathrm{L}}^{(2)}=\frac{\mathrm{i}2I^2\psi_0^2}{(2\pi)^7\hbar\beta_1}\left\{\frac{1}{15}\left(\frac{7}{\alpha_2^5}-\frac{9}{\alpha_2^4\alpha_1}+\frac{2}{\alpha_2^3\alpha_1^2}+\frac{2}{\alpha_2^2\alpha_1^3}-\frac{9}{\alpha_2\alpha_1^4}+\frac{7}{\alpha_1^5}\right)\right.$$
$$+\frac{2}{\alpha_1^5}\ln\left(\frac{\alpha_1+\alpha_2}{2\alpha_2}\right)+\frac{2}{\alpha_2^5}\ln\left(\frac{\alpha_1+\alpha_2}{2\alpha_1}\right)+\frac{1}{5}\left(\frac{1}{2}-\frac{\alpha_2}{\alpha_1+\alpha_2}\right)\left(\frac{1}{\alpha_2^5}-\frac{1}{\alpha_1^5}\right)$$
$$+\frac{1}{21}(p_1^2+p_2^2+p_1p_2\cos\theta_p)\left[-\frac{1}{\alpha_2^7}\ln\left(\frac{4\alpha_1\alpha_2}{(\alpha_1+\alpha_2)^2}\right)\right.$$
$$-\frac{47}{10\alpha_2^7}+\frac{5}{\alpha_2^6\alpha_1}-\frac{9}{5\alpha_2^5\alpha_1^2}+\frac{5}{4\alpha_2^4\alpha_1^3}+\frac{5}{4\alpha_2^3\alpha_1^4}$$
$$-\frac{9}{5\alpha_2^2\alpha_1^5}+\frac{5}{\alpha_2\alpha_1^6}-\frac{47}{10\alpha_1^7}-\left(\frac{1}{2}-\frac{\alpha_2}{\alpha_1+\alpha_2}\right)\left(\frac{1}{\alpha_2^7}-\frac{1}{\alpha_1^7}\right)$$
$$\left.\left.+\frac{3}{2}\left(\frac{1}{\alpha_2^7}\ln\frac{2\alpha_1}{\alpha_1+\alpha_2}+\frac{1}{\alpha_1^7}\ln\left(\frac{2\alpha_2}{\alpha_1+\alpha_2}\right)\right)\right]\right\}. \tag{24f}$$

This result has qualitative similarities with equation (22b), since the Landau parameters F_1 and F_2 are imaginary and small for the case of large spatial damping (α_1 and α_2 large). This supports the conclusion that Γ_{L} is an inadequate representation of the effective interaction in ^{3}He if the bare interaction is characterised by very large, but finite, damping.

In addition to the above cases, it has been possible to evaluate $\gamma(\xi)$ for $\psi(p) = p^{\nu}\exp(-\beta_2 p)$, for which

$$V(\boldsymbol{r}, t) = \frac{I\nu!}{(2\pi)^4(\beta_2 + r)^{\nu+1}}\exp(-\beta_1|t|)$$

and $\psi(p) = p^3\cos(\beta_2 p)\exp(-\varepsilon p)$, for which

$$V(\boldsymbol{r}, t) = \frac{6I}{(2\pi)^4}\frac{(r+\varepsilon)^4 - 6(r+\varepsilon)^2\beta_2^2 + \beta_2^4}{[(r+\varepsilon)^2 + \beta_2^2]^4}\exp(-\beta_1|t|).$$

In both cases we obtain lengthy sums for $\gamma(\xi)$ involving Whittaker functions, which will be omitted for the sake of brevity. These two cases, together with example 2, show that it is not necessary for applicability of the present method that $V(\boldsymbol{r}, t)$ have an exponential r dependence, while example 3 shows how we may include both attractive and repulsive forces.

4. Summary and conclusion

The object of the foregoing sections has been to extend a previous work (Nettleton 1979) which discussed the solution of the Bethe–Salpeter equation for Γ_L, the ladder approximation to the effective interaction in a Fermi liquid, using a procedure for converting the integral equation to differential form. This approach was originally devised by Hahne *et al* (1977) for the case of interactions $V(t)$ depending only on time and not on space. It proved possible to extend this (Nettleton 1979) to the case of a spatial dependence for $V(\boldsymbol{r}, t)$ proportional to $\delta(\boldsymbol{r})$, but this led to a contour integration in q space which was difficult to evaluate. In the present paper, we show that the same mathematical difficulties do not occur when we use a bare potential having finite range. Specifically, we consider interactions whose r dependence can be cast in the form of a Laplace transform, and thus the method can be applied, in principle, to any bare particle potential whose inverse Laplace transform exists.

It is instructive to compare the results we obtain for Γ_L using soft potentials with the effective interaction calculated in the ladder approximation from the contact interaction of equation (1). The latter is real and depends on $|\boldsymbol{p}_2 - \boldsymbol{p}_1|$. Equations (13), (14), and (20), on the other hand, are all complex and lose their momentum dependence as $\beta_1 \to \infty$ or $\beta_2 \to \infty$, which are the limits in which the soft potential approaches V_c when $I = O(\beta_1\beta_2^3)$. It is thus evident that a change in the bare potential from zero-range to very short range, which may seem physically insignificant, can have a marked effect on the analytical character of the ladder sum.

The fact that equation (1) led to a Γ_L peaked at $|\boldsymbol{p}_1 - \boldsymbol{p}_2| = 0$ suggested that Γ_L be interpreted as a scattering amplitude (Levin and Valls 1978). However, the absence of pronounced momentum dependence in equation (22*b*) leaves us unable to assert that the ladder approximation is dominant in either the scattering amplitude or effective interaction, since the same particle–hole diagrams contribute to both. Therefore we are led to agree with other authors (Levin and Valls 1979) that diagram summation will probably not yield a reliable *ansatz* for the quasi-particle interaction in ^{3}He.

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